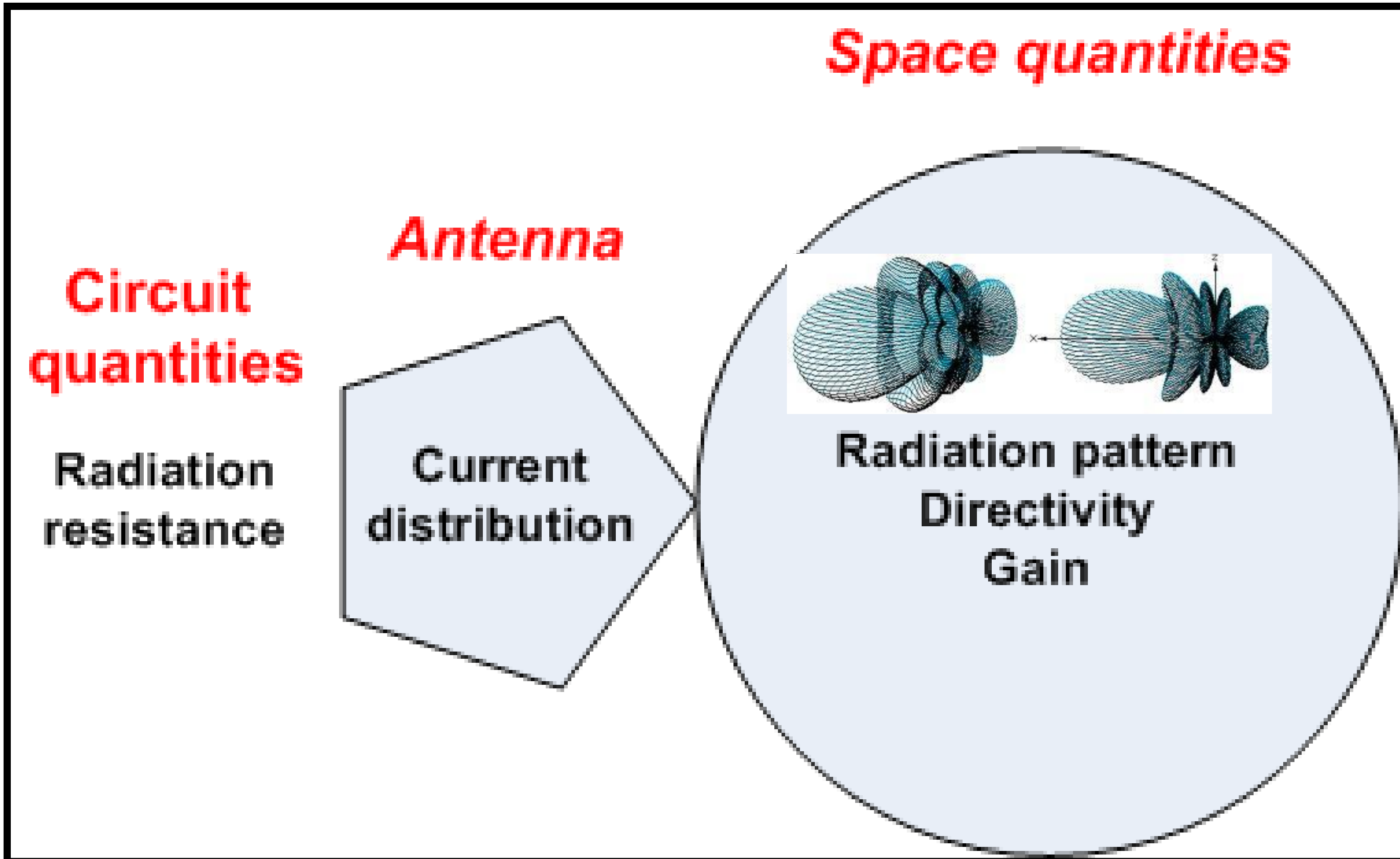
Lecture 4


Space quantities
Antenna
Circuit quantities
Radiation resistance
Current distribution
Radiation pattern
Directivity
Gain

Maxwell's equations

The physics of the fields radiated by an antenna are described by Maxwell's equations. For harmonic variations of the fields $(e^{j\omega t})$, we can write

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \vec{J} \qquad (1)$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad (2)$$

$$\nabla \cdot \vec{D} = \varepsilon\,\nabla \cdot \vec{E} = \rho \qquad (3)$$

$$\nabla \cdot \vec{B} = \mu\,\nabla \cdot \vec{H} = 0 \qquad (4)$$

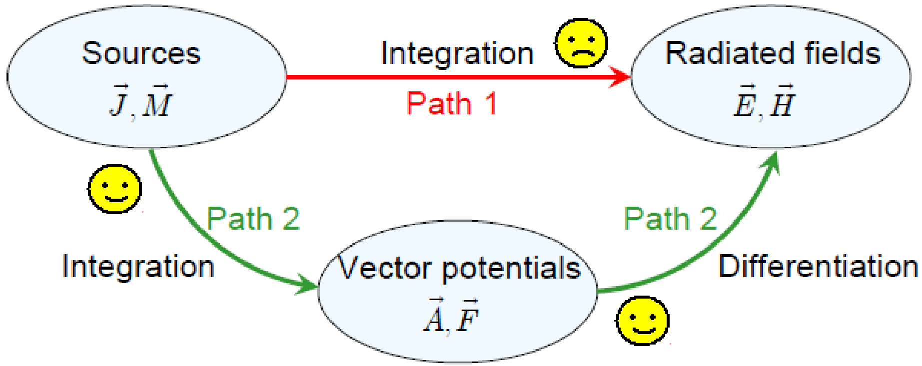
with $\vec{J} \begin{cases} \neq 0 & \text{in source region} \\ = 0 & \text{elsewhere} \end{cases}$

E, H: Electric and magnetic field
D: Dielectric displacement
B: Magnetic flux (induction)
J: Electric source current density
ρ: Charge density

$\nabla \times \vec{V} = \mathrm{curl}\,\vec{V}$
$\nabla \cdot \vec{V} = \mathrm{div}\,\vec{V}$
$\nabla \phi = \mathrm{grad}\,\phi$

Vector potentials

To analyze the fields radiated by sources, it is common practice to introduce auxiliary functions known as **vector potentials**, which will aid in the solution of Maxwell's equations.



- The two-step procedure usually involves simpler integrations than the direct path.
- The use of vector potentials basically permits to reduce the number of unknowns.

div curl = 0

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{a}}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{a}}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{a}}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\nabla \cdot \nabla \times \mathbf{A} = \frac{\partial .}{\partial x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \frac{\partial .}{\partial z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) = 0$$

Reciprocally,

If the divergence of a vector field equals zero, then there exists a potential vector field so that the curl of the potential field equals the vector field

$\nabla \cdot \vec{V} = 0 \quad \Rightarrow$ there exists $\vec{U}$ so that $\vec{V} = \nabla \times \vec{U}$ (*)

curl grad $= 0$

$$\nabla\psi = \hat{\mathbf{a}}_x \frac{\partial \psi}{\partial x} + \hat{\mathbf{a}}_y \frac{\partial \psi}{\partial y} + \hat{\mathbf{a}}_z \frac{\partial \psi}{\partial z}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{a}}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{a}}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{a}}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\nabla \times \nabla\psi = \hat{\mathbf{a}}_x \left(\frac{\partial}{\partial y} \frac{\partial \psi}{\partial z} - \frac{\partial}{\partial z} \frac{\partial \psi}{\partial y} \right)$$

$$+ \hat{\mathbf{a}}_y \left(\frac{\partial}{\partial z} \frac{\partial \psi}{\partial x} - \frac{\partial}{\partial x} \frac{\partial \psi}{\partial z} \right)$$

$$+ \hat{\mathbf{a}}_z \left(\frac{\partial}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial}{\partial y} \frac{\partial \psi}{\partial x} \right) = 0$$

Reciprocally,

If the curl of a vector field equals zero, then this vector field can be written as the gradient of a potential function

$\nabla \times \vec{V} = 0 \quad \Rightarrow$ there exists ϕ so that $\vec{V} = \nabla\phi$ (**)

Vector potential $\vec{A}$ for electric current source $\vec{J}$

Since $\nabla \cdot \vec{B} = 0$, the magnetic flux $\vec{B}$ can be represented as the curl of another vector $\vec{A}$

$$\vec{B}_A = \nabla \times \vec{A} \longrightarrow \boxed{\vec{H}_A = \frac{1}{\mu}\nabla \times \vec{A}}$$

The vector $\vec{A}$ is called ***magnetic vector potential***.

$$\nabla \times \vec{E}_A = -j\omega\mu\vec{H}_A = -j\omega\nabla \times \vec{A} \longrightarrow \boxed{\nabla \times \left(\vec{E}_A + j\omega\vec{A}\right) = 0}$$

define the ***electric scalar potential*** ϕ_e

$(\nabla \times (-\nabla\phi_e) = 0)$

Since $\nabla \times \left(\vec{E}_A + j\omega\vec{A}\right) = 0$, there exists a scalar function ϕ_e so that

$$\vec{E}_A + j\omega\vec{A} = -\nabla\phi_e \longrightarrow \boxed{\vec{E}_A = -\nabla\phi_e - j\omega\vec{A}}$$

$$\nabla \times \vec{H}_A = j\omega\varepsilon \vec{E}_A + \vec{J}$$ & $$\vec{H}_A = \frac{1}{\mu}\nabla \times \vec{A}$$ $$\vec{E}_A = -\nabla\phi_e - j\omega\vec{A}$$

The vector potential $\vec{A}$ is defined through its curl.

$$\frac{1}{\mu}\nabla \times \nabla \times \vec{A} = j\omega\varepsilon(-\nabla\phi_e - j\omega\vec{A}) + \vec{J}$$

$$\because \quad \nabla \times \nabla \times \vec{A} = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2\vec{A}$$

$$\nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2\vec{A} = -j\omega\varepsilon\mu\,\nabla\phi_e + \omega^2\varepsilon\mu\vec{A} + \mu\vec{J}$$

Lorenz condition define The vector potential $\vec{A}$ divergence. → $\nabla \cdot \vec{A} = -j\omega\varepsilon\mu\phi_e$

$$\nabla^2\vec{A} + \omega^2\varepsilon\mu\vec{A} = -\mu\vec{J}$$ Inhomogeneous wave equation for $\vec{A}$

The electromagnetic wave equation is a second-order partial differential equation that describes the propagation of electromagnetic waves through a medium or in a vacuum

SOLUTION OF THE INHOMOGENEOUS VECTOR POTENTIAL WAVE EQUATION

assume source *Jz (directed along z axis)* is an infinitesimal source

$$\nabla^2 A_z + k^2 A_z = -\mu J_z$$

the source is a point (infinitesimal) so *Az* is not a function of direction (θ and ϕ)

a. Assume free of source i.e homogeneous equation will be

$$\nabla^2 A_z + k^2 A_z = 0 \longrightarrow \nabla^2 A_z(r) + k^2 A_z(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial A_z(r)}{\partial r}\right] + k^2 A_z(r) = 0$$

$$\frac{d^2 A_z(r)}{dr^2} + \frac{2}{r}\frac{dA_z(r)}{dr} + k^2 A_z(r) = 0$$

differential equation has two independent solutions

$$A_{z1} = C_1 \frac{e^{-jkr}}{r}$$

outwardly traveling wave ✓

$$A_{z2} = C_2 \frac{e^{+jkr}}{r}$$

inwardly traveling wave

$$\nabla\psi = \hat{\mathbf{a}}_r \frac{\partial \psi}{\partial r} + \hat{\mathbf{a}}_\theta \frac{1}{r}\frac{\partial \psi}{\partial \theta} + \hat{\mathbf{a}}_\phi \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}$$

$$\nabla \cdot \nabla\psi = \nabla^2\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

$$\nabla^2 \mathbf{A} = \hat{\mathbf{a}}_x \nabla^2 A_x + \hat{\mathbf{a}}_y \nabla^2 A_y + \hat{\mathbf{a}}_z \nabla^2 A_z$$

b. Assume no propagation k = 0 and source ($J_z \neq 0$) the wave equation

$$\nabla^2 A_z + k^2 A_z = -\mu J_z \longrightarrow \nabla^2 A_z = -\mu J_z$$

→ *It is recognized to be Poisson's equation whose solution is widely documented*

solution

$$A_z = \frac{\mu}{4\pi} \iiint_V \frac{J_z}{r}\, dv'$$

Final solution from a. and b.

$$A_z = \frac{\mu}{4\pi} \iiint_V J_z \frac{e^{-jkr}}{r}\, dv'$$

For electric current I_e *on wire antennas final solution* reduce to line integrals

$$\mathbf{A} = \frac{\mu}{4\pi} \int \mathbf{I}_e(x', y', z') \frac{e^{-jkR}}{R}\, dl'$$

Note:

A_x in same direction of I_x
A_y in same direction of I_y
A_z in same direction of I_z

SUMMARY

electromagnetic fields can be calculated from single vector potential A

1- **Specify J and find A from**

$$\vec{A} = \frac{\mu}{4\pi} \iiint_V \vec{J} \frac{e^{-jkR}}{R} dv'$$

2- **From A the fields H and E can be determined from**

$$\vec{H}_A = \frac{1}{\mu} \nabla \times \vec{A} \longrightarrow \vec{E}_A = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}_A$$

Infinitesimal Dipole $(\ell \leq \lambda / 50)$

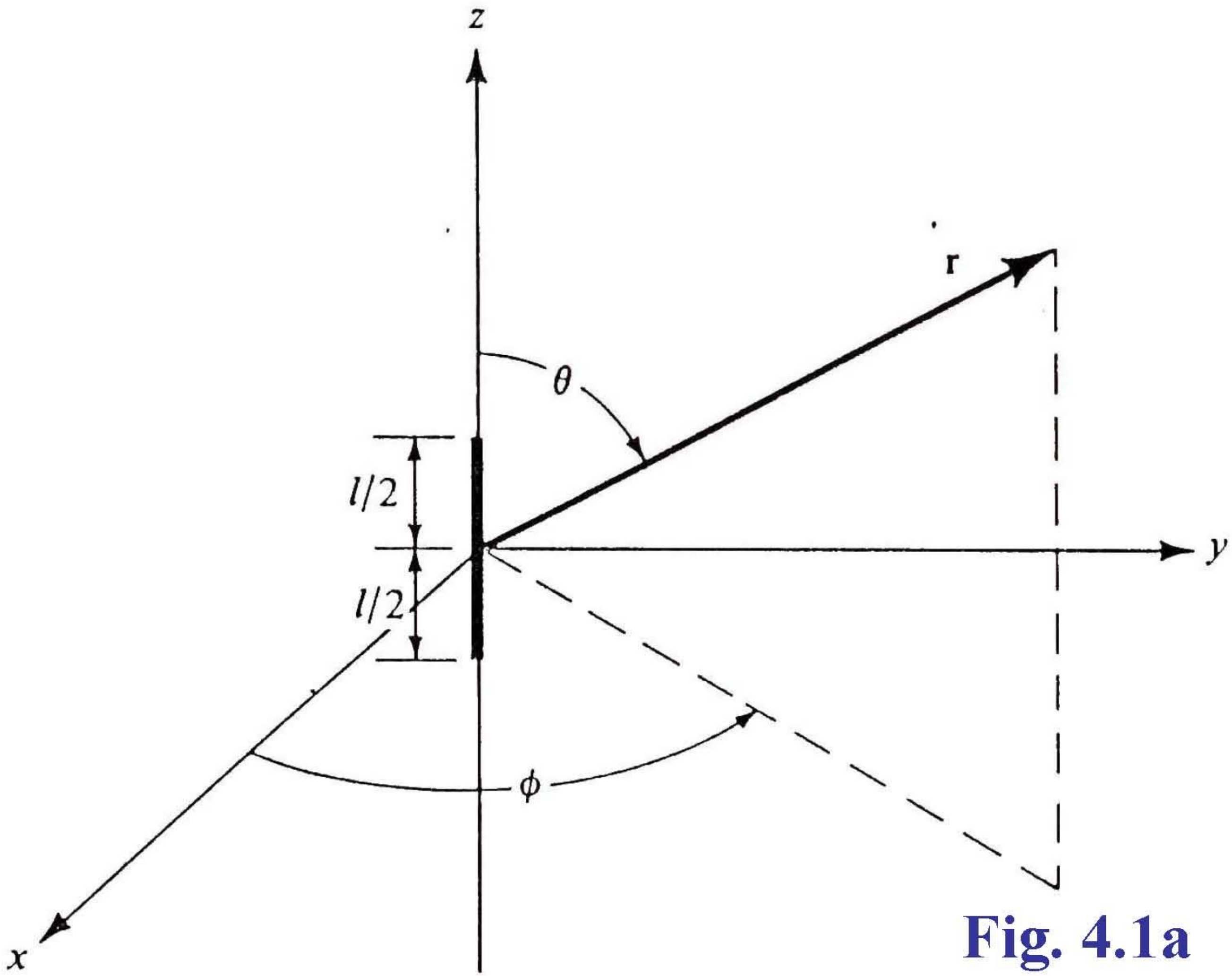


Fig. 4.1a

Infinitesimal Dipole

- An infinitesimal linear wire ($l \ll \lambda$) is oriented along the z axis.

$$\boldsymbol{I}(z') = \hat{\boldsymbol{a}}_z I_0$$

$$\boldsymbol{A}(x,y,z) = \frac{\mu}{4\pi}\int_C \boldsymbol{I}_e(x',y',z')\frac{e^{-jkR}}{R}\,dl'.$$

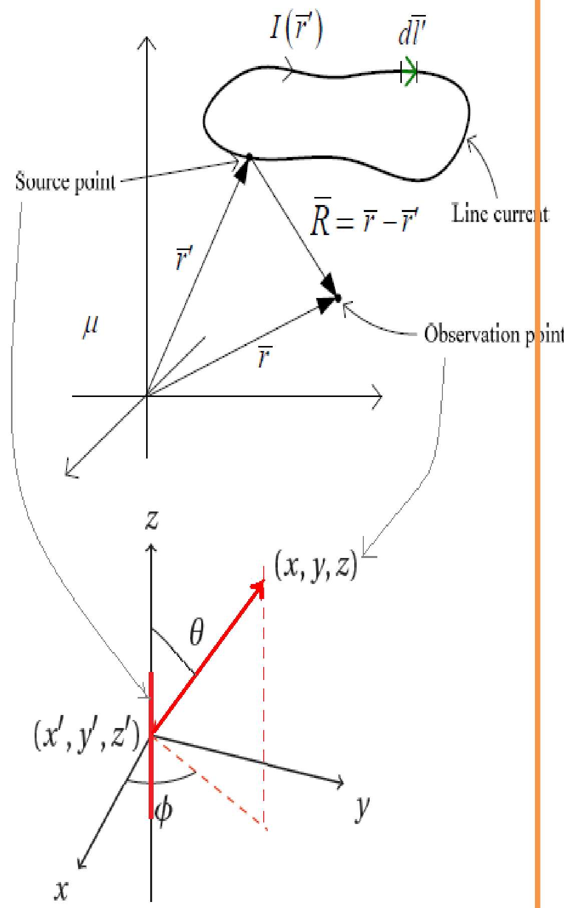
where

(x,y,z) : Observation point coordinates.

(x',y',z') : Coordinates of the source.

R : The distance from any point on the source to the observation point.

C : Path along the length of the source.



$$\boldsymbol{A}(x,y,z) = \hat{\boldsymbol{a}}_z \frac{\mu I_0}{4\pi r} e^{-jkr}\int_{-l/2}^{l/2} dz' = \hat{\boldsymbol{a}}_z \frac{\mu I_0 l}{4\pi r} e^{-jkr}.$$

$$\boldsymbol{H}_A = \frac{1}{\mu}\nabla \times \boldsymbol{A} \qquad\qquad \nabla \times \boldsymbol{H}_A = \boldsymbol{J} + j\omega\epsilon \boldsymbol{E}_A\ .$$

Transformation from rectangular to spherical coordinates:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$A_x = 0, A_y = 0, A_z \neq 0.$

$$A_r = A_z\cos\theta = \frac{\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta.$$

$$A_\theta = -A_z\sin\theta = -\frac{\mu I_0 l e^{-jkr}}{4\pi r}\sin\theta.$$

$$A_\phi = 0.$$

$$\nabla \times \vec{A} = \frac{1}{r^2 \, sin\theta} \begin{bmatrix} \hat{a}_r & r\hat{a}_\theta & rsin\theta \hat{a}_\emptyset \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \emptyset} \\ A_r & rA_\theta & rsin\theta A_\emptyset \end{bmatrix}$$

$$\boldsymbol{H} = \frac{1}{\mu}\frac{\hat{\boldsymbol{a}}_\phi}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial}{\partial \theta} A_r\right]$$

$$H_r = H_\theta = 0.$$

$$H_\phi = j\frac{kI_0 l \sin\theta}{4\pi r}\left[1 + \frac{1}{jkr}\right]e^{-jkr}.$$

The Electric field generated from magnetic field in free space, thus J=0 and thus E and H components are valid every where except on the source itself

$$E = E_A = \frac{1}{j\omega\epsilon}\nabla \times H.$$

$$E_r = \eta\frac{I_0 l\cos\theta}{2\pi r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr}.$$

$$E_\theta = j\eta\frac{kI_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}.$$

$$E_\phi = 0.$$

Far-Field ($kr \gg 1$) Region

$$E_\theta \simeq j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \sin\theta$$

$$E_r \simeq E_\phi = H_r = H_\theta = 0$$

$$H_\phi \simeq j \frac{kI_0 l e^{-jkr}}{4\pi r} \sin\theta$$

The ratio of E_θ to H_ϕ is equal to

$$\frac{E_\theta}{H_\phi} \simeq \eta$$

Far-Field Region

$$\vec{E}_A = -\nabla\phi_e - j\omega\vec{A}$$

$$\left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -j\omega A_\theta \\ E_\phi &\simeq -j\omega A_\phi \end{aligned}\right\} \Rightarrow \boxed{\mathbf{E}_A \simeq -j\omega\mathbf{A}}$$

(for the θ and ϕ components only since $E_r \simeq 0$)

$$A_r = A_z \cos\theta = \frac{\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta$$

$$A_\theta = -A_z \sin\theta = -\frac{\mu I_0 l e^{-jkr}}{4\pi r}\sin\theta$$

$$A_\phi = 0$$

$$E_\theta \simeq -j\omega A_\theta = j\omega\frac{\mu I_0 l e^{-jkr}}{4\pi r}\sin\theta = j\eta\frac{kI_0 l \sin\theta}{4\pi r}e^{-jkr}$$